

Space-time trigonometry and formalization of the “Twin Paradox” for uniform and accelerated motions

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Abstract - The formal structure of the early Einstein’s Special Relativity follows the axiomatic deductive method of Euclidean geometry. In this paper we show the deep-rooted relation between Euclidean and space-time geometries that are both linked to a two-dimensional number system: the complex and hyperbolic numbers, respectively.

By studying the properties of these numbers together, pseudo-Euclidean trigonometry has been formalized with an axiomatic deductive method and this allows us to give a complete quantitative formalization of the twin paradox in a familiar “Euclidean” way for uniform motions as well as for accelerated ones.

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1 Introduction

The final part of § 4 of the famous Einstein's 1905 special relativity paper [1] contains the sentences concerning moving clocks on which volumes have been written: “.. If we assume that the result proved for a polygonal line is also valid for a continuously curved line, we obtain the theorem: If one of two synchronous clocks at A is moved in a closed curve with constant velocity until it returns to A, the journey lasting t seconds, then the clock that moved runs $\frac{1}{2} t \left(\frac{v}{c}\right)^2$ seconds [2] slower than the one that remained at rest”.

About six years later, on 10 April 1911, at the Philosophy Congress at Bologna, Paul Langevin replaced the clocks A and B with human observers and “twin paradox” officially was born.

Langevin, using the example of a space traveller who travels a distance L (measured by someone at rest on the earth) in a straight line to a star in one year and than abruptly turns around and returns on the same line, wrote: “..Revenu à la Terre ayant vielli deux ans, il sortira de son arche et trouvera notre globe vielli deux cents ans si sa vitesse est restée dans l'interval inférieure d'un vingt-millième seulement à la vitesse de la lumière.” [3] We must remark that Langevin, besides not rejecting ether's existence, stresses the point which will be the subject of the subsequent discussions, that is the asymmetry between the two reference frames.

The space traveller undergoes an acceleration halfway of his journey, while the twin at rest in the earth reference frame always remains in an inertial frame.

For Langevin, every acceleration has an absolute meaning. Even though the effect foreseen by Einstein's theory has got several experimental confirmations, the contribution of accelerated stretches of the path still stands as a subject of discussion and controversies.

Aiming to not causing misunderstandings, we stress that the discussions we allude are rigorously confined to the ambit of special relativity, that is to the space-time of special relativity.

It is in this ambit that Rindler says: “...If an ideal clock moves *nonuniformly* through an inertial frame, we shall *assume* that acceleration as such has no effect on the rate of the clock, i.e., that its instantaneous rate depends only on its instantaneous speed v according to the above rule. Unfortunately, there is no way of *proving* this. Various effects of acceleration on a clock would be consistent with S. R. Our assumption is one of simplicity - or it can be regarded as a definition of an “ideal” clock. We shall call it the clock hypothesis.” [4].

We think that a conclusion on the role of the accelerated motions and, most of all, an evaluation of the amount of the slowing down of an accelerated clock can only be reached through a rigorous and exhaustive exploitation of the mathematics of special relativity.

If the theory has no logical inconsistencies, the theory itself must thereby provide a completely accurate account of the asymmetrical aging process.

Even if Minkowski gave a geometrical interpretation of the special relativity space-time shortly after (1907-1908) Einstein's fundamental paper, a mathematical tool exploitable in the context of the Minkowski space-time has begun to be carried out only few decades ago ([5]-[10]).

This mathematical tool is based on the use of hyperbolic numbers, introduced by S. Lie in the late XIX century [11].

In analogy with the procedures applied in the case of complex numbers, it is possible to formalize, also for the hyperbolic numbers, a space-time geometry and a trigonometry following the same Euclidean axiomatic-deductive method [10, 12].

In this paper, we first summarize the introduction of the hyperbolic space-time trigonometry and then apply it to formalize the twin paradox for inertial motions as well as for accelerated ones.

The self-consistency of the method allows us to solve any problem in the Minkowski space-time through an elementary approach as if we were working on Euclidean plane.

We would conclude this introduction with an epistemological consideration which, in the centenary

of special relativity, turns out to be a recognition of the Einstein's insight.

We know that Einstein formalized special relativity by starting from two axioms and applying the axiomatic-deductive method of Euclidean geometry [1].

Euclid's geometry and special relativity are both associated with group theory and, in the same way, to complex and hyperbolic numbers [8] at the extent that we can say: Euclidean geometry and the geometry of Minkowski space-time are both deriving from a second degree algebraic equation [5, 13]. In particular:

1. from square roots of negative quantities we have complex numbers and Euclidean geometry;
2. from square roots of positive quantities we have hyperbolic numbers and Minkowski space-time geometry.

Then these two geometries have a common source and for this reason can be considered as equivalent. Perhaps the "ingenious intuition" of the general laws of nature let him guess an incredible equivalence in spite of the apparent differences.

2 Hyperbolic trigonometry

2.1 Basic definitions

Complex numbers are strictly related to Euclidean geometry: indeed their invariant (the modulus) is the same as the Pythagoric distance (Euclidean invariant) and their unimodular multiplicative group is the Euclidean rotation group. As it is known these properties allow us to use complex numbers for representing plane vectors.

In the same way hyperbolic numbers, an extension of complex numbers [5, 14] defined as

$$\{z = t + hx; h^2 = 1, t, x \in \mathbf{R}; h \notin \mathbf{R}\},$$

are strictly related to space-time geometry [8, 10]. Indeed their modulus is given by (we call $\tilde{z} = t - hx$ the hyperbolic conjugate of z as for complex numbers)

$$|z|^2 = z\tilde{z} \equiv t^2 - x^2, \tag{1}$$

and if t is given the physical meaning of a normalized time variable (the speed of light $c = 1$) and x the meaning of a space variable, then Eq. (1) is the Lorentz invariant of two dimensional special relativity [6]. Moreover their unimodular multiplicative group is the special relativity Lorentz's group [6, 8].

Then hyperbolic numbers represent for space-time plane the same complex numbers represent for Euclidean plane. Thanks to this correspondence and by pointing out the analogies and the differences with these two number systems, the space-time trigonometry has been formalized with the same rigour as Euclidean one [10], [12].

In fact the theorems of Euclidean trigonometry are usually obtained through elementary geometry observations. Otherwise we can define in a Cartesian plane the trigonometric functions directly from Euclid's rotation group (as shown in the appendix A.1) and, as a consequence, the trigonometry theorems will follow just as mathematical identities. Now since we know that the Lorentz group enjoys, for space-time geometry, the same invariance property as the rotation group does for Euclidean geometry, we can introduce in a Cartesian plane the hyperbolic trigonometric functions through the properties of Lorentz group described by hyperbolic numbers [10]. The importance of this introduction relies on the fact that we do not have for pseudo-Euclidean geometry the same intuitive vision we have for Euclidean geometry. The obtained results and their complete coherence will provide a Euclidean picture of pseudo-Euclidean geometry.

This picture can be considered analogous to the representation, on a Cartesian plane, of the surface differential geometry by means of the distance between two points given by the Lorentz invariant (space-time interval or proper time), instead of the Pythagorean one. In this plane the Lorentz transformations (uniform motions) are represented by straight-lines, and the curved lines represent non-uniform (accelerated) motions [8]. In particular the constant accelerated motions are given by an arm of equilateral hyperbola. [15, 16]

Here we briefly summarize some fundamental properties of hyperbolic numbers.

This number system has been introduced by S. Lie [11] as a two dimensional example of the more general class of the commutative hypercomplex numbers systems [13].

Now let us introduce a hyperbolic plane on the analogy of the Gauss-Argand plane of the complex variable. In this plane we associate the point $P \equiv (t, x)$ to the hyperbolic number $z = t + hx$. If we represent this number on a Cartesian plane, in this plane the square of z distance (D^2) from the origin of the coordinate axes is defined as

$$D^2 = z \tilde{z} \equiv t^2 - x^2. \quad (2)$$

Let us consider the multiplicative inverse of z that, if existing, is given by: $1/z \equiv \tilde{z}/z\tilde{z}$. This implies that z does not have an inverse when $z\tilde{z} \equiv t^2 - x^2 = 0$, i.e., when $x = \pm t$, or alternatively when $z = t \pm ht$. These two straight-lines in the hyperbolic plane, whose elements have no inverses, divide the hyperbolic plane in four sectors that can be called *Right sector (Rs)*, *Up sector (Us)*, *Left sector (Ls)*, and *Down sector (Ds)*. This property is the same as that of the Minkowski plane and this correspondence assigns the physical meaning of **proper time** (space-time interval) to the definition of distance [15]. Let us now consider the quantity $t^2 - x^2$, which is positive in the *Rs, Ls* ($|t| > |x|$) sectors, and negative in the *Us, Ds* ($|t| < |x|$) sectors. This quantity, as known from special relativity, must have its sign and appear in this quadratic form. In particular, in the case we had to use the linear form $\sqrt{t^2 - x^2}$, (the modulus of hyperbolic numbers, or the triangle side length), we will follow the definition of Yaglom ([5] p. 180) and Chabat ([14] p. 51), and take the absolute value of argument $\sqrt{|t^2 - x^2|}$.

Now let us introduce the hyperbolic exponential function and hyperbolic polar transformation. The hyperbolic exponential function in pseudo-Euclidean geometry plays the same important role as the complex exponential function in Euclidean geometry. Comparing absolutely convergent series it can be written [6, 14]

for $|t| > |x|$, $t > 0$ (i.e., $t, x \in Rs$)

$$t + hx = \exp[\rho' + h\theta] \equiv \exp[\rho'](\cosh \theta + h \sinh \theta) \quad (3)$$

The exponential function allows us to introduce the **hyperbolic polar transformation**.

Following [6, 14] we define the *radial coordinate* as

$$\exp[\rho'] \Rightarrow \rho = \sqrt{t^2 - x^2}$$

and *the angular coordinate* as

$$\theta = \tanh^{-1}(x/t) \equiv \tanh^{-1} v$$

Then the hyperbolic polar transformation is defined as

$$t + hx \Leftrightarrow \rho \exp[h\theta] \equiv \rho(\cosh \theta + h \sinh \theta). \quad (4)$$

Given two points $P_j \equiv z_j \equiv (x_j, y_j)$, $P_k \equiv z_k \equiv (x_k, y_k)$ we define the “*square distance*” between them by extending Eq. (2)

$$D_{j,k} = (z_j - z_k)(\tilde{z}_j - \tilde{z}_k). \quad (5)$$

As a general rule we indicate the square of the segment lengths by capital letters, and by the same small letters the square root of their absolute value [5, 14]

$$d_{j,k} = \sqrt{|D_{j,k}|}. \quad (6)$$

Following the usual convention [15], a segment or line is said to be *timelike* (*spacelike*) if it is parallel to a line through the origin located in the sectors containing the axis t (x). Then the segment $\overline{P_j P_k}$ is time-like (space-like) if $D_{j,k} > 0$ ($D_{j,k} < 0$), and *lightlike* or null lines if $D_{j,k} = 0$.

2.2 Basic relations in the pseudo-Euclidean triangles

The guide-lines for “Euclidean” formalization of space-time trigonometry are summarized in appendix A. An exhaustive treatment of this subject can be found in [10]. Here we only report the conclusions which allow us to formalize the twin paradox.

As a matter of fact the same laws that hold for Euclidean trigonometry are true for pseudo-Euclidean trigonometry and the latter can be obtained from the former by means of the following substitutions [10]

1) Euclidean distance $x^2 + y^2 \Rightarrow$ pseudo-Euclidean distance $t^2 - x^2$

2) Circular angles \Rightarrow hyperbolic angles.

3) The straight-lines equations are expressed by means of hyperbolic trigonometric functions [17].

Moreover, all the theorems that hold in Euclidean geometry for the circle (invariant curve [5]) are changed in the same ones for equilateral hyperbolas (invariant curve for pseudo-Euclidean plane [6]).

In particular, the equation of an equilateral hyperbola depends on three conditions, that can be the same we require to determine a circle in a Cartesian plane.

As a function of the center coordinates (t_C, x_C) and the diameter ($2p$), it is given by

$$(t - t_C)^2 - (x - x_C)^2 = p^2 \quad (7)$$

or, in parametric form, by Eq. (16) of the next section.

Now we recall the theorems that will be used in this paper. We call ρ_i , for $i = 1, 2, 3$, the lengths of three sides of a triangle, θ_i the opposite angles to ρ_i , $2p$ the “diameter” of the equilateral hyperbola “circumscribed” to the triangle; we have

• *Law of sines.*

$$\frac{\rho_1}{\sinh_e \theta_1} = \frac{\rho_2}{\sinh_e \theta_2} = \frac{\rho_3}{\sinh_e \theta_3} = 2p. \quad (8)$$

• *Second law of cosines.*

$$\rho_i = |\rho_j \cosh_e \theta_k + \rho_k \cosh_e \theta_j|. \quad (9)$$

• *The sum of the internal angles in a triangle satisfies the same relations as in Euclidean triangles [18]*

$$\sinh_e(\theta_1 + \theta_2 + \theta_3) = 0, \quad \cosh_e(\theta_1 + \theta_2 + \theta_3) = -1. \quad (10)$$

• *If we have points A and B on the same arm of an equilateral hyperbola, for any point P outside arc AB , hyperbolic angles \widehat{APB} are the same. If we call C the center of the equilateral hyperbola, we also have $\widehat{ACB} = 2\widehat{APB}$.*

From a mathematical point of view, with the extension of the trigonometric hyperbolic functions exposed in [6], [10] and summarized in appendix A.2, the hyperbolic trigonometry holds in the whole hyperbolic plane and allows us to consider triangles having sides in whatever direction [10][19]. As far as this paper is concerned we are dealing with physical phenomena that are represented just in

(Rs) [15], and the hyperbolic functions are the classical ones, taking into account that the hyperbolic angles can be measured, in particular for the parametric form of equilateral hyperbola, with respect to a straight-line parallel to the x axis.

Moreover, since the above summary may result inadequate to make the reader familiar with hyperbolic trigonometry we preferred, in some examples, not to use directly the hyperbolic counterpart of Euclidean theorems. We obtain the results by means of simple mathematics and afterwards show the mentioned correspondence.

3 Mathematical formalization of the twin paradox

As we have already emphasized in the introduction, a consequence of the Lorentz transformations is the so called “Twin paradox”. After a century this problem continues to be the subject of many papers, not only relative to experimental tests [20] but also regarding physical and philosophical considerations [21].

In this paper we want to show how the formalization of hyperbolic trigonometry [10] allows us, with elementary mathematics, a formalization of this problem both for uniform and accelerated motions.

3.1 Inertial motions

In a representative t, x plane let us start with the following example: a twin is steady in the point $x = 0$, his path is represented by the t axis. The other twin, on a rocket, starts with speed v from $O \equiv (0, 0)$ and after a time τ_1 , at the point T , he reverses its direction and comes back arriving to the point $R \equiv (\tau_2, 0)$ [22]. In Fig. 1 we represent this problem by means of the triangle OTR .

From a geometrical point of view we can compare the elapsed times for the twins by comparing the “lengths” (proper times) of the sum $\overline{OT} + \overline{TR}$ and of the side \overline{OR} .

The qualitative interpretation is reported in many books and is easily explained by means of the reverse triangle inequality in space-time geometry with respect to Euclidean geometry ([23] p. 130). Also a graphical visualization can be easily performed considering that a segment must be reported on another by means of an equilateral hyperbola, instead of Euclidean circle ([5] p. 190).

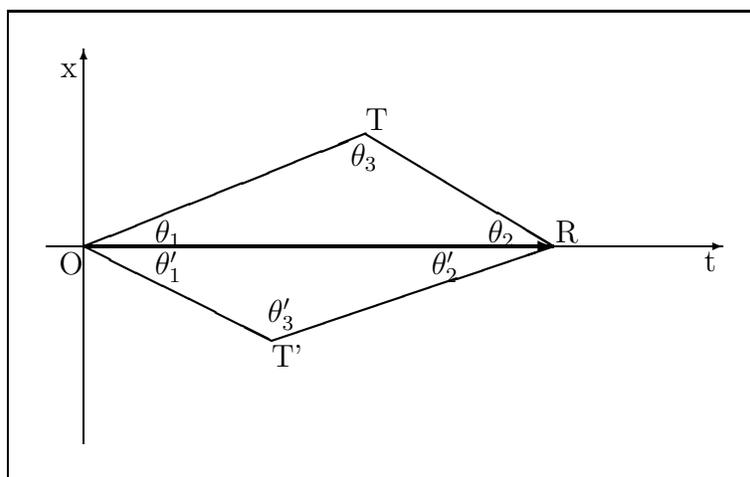


Figure 1: The twin paradox for uniform motions

Now we will see that Euclidean formalization of space-time trigonometry [10] allows us to obtain a simple quantitative formulation of the problem.

Let us call $\theta_1 \equiv \tanh v$ the hyperbolic angle \widehat{ROT} , θ_2 the hyperbolic angle \widehat{ORT} and θ_3 the hyperbolic angle \widehat{OTR} . From their physical meaning the angles θ_1 and θ_2 are so that the straight-lines \overline{OT} and \overline{TR} are time-like [15] (in a Euclidean representation the angle of the straight-lines with the t axis must be less than $\pi/4$).

Let us apply to the side \overline{OR} the second cosine law (9); we have

$$\overline{OR} = \overline{OT} \cosh \theta_1 + \overline{TR} \cosh \theta_2. \quad (11)$$

It follows that the difference between the twins' proper times $\Delta\tau$ is

$$\Delta\tau \equiv \overline{OR} - \overline{OT} - \overline{TR} = \overline{OT} (\cosh \theta_1 - 1) + \overline{TR} (\cosh \theta_2 - 1). \quad (12)$$

If we call p the semi-diameter of the equilateral hyperbola circumscribed to the triangle \widehat{OTR} , from Eq. (8) we have $\overline{OT} = 2p \sinh \theta_2$; $\overline{TR} = 2p \sinh \theta_1$, and

$$\Delta\tau = 2p (\cosh \theta_1 \sinh \theta_2 + \cosh \theta_2 \sinh \theta_1 - \sinh \theta_1 - \sinh \theta_2) \equiv 2p [\sinh(\theta_1 + \theta_2) - \sinh \theta_1 - \sinh \theta_2]. \quad (13)$$

Now we can consider the following problem: given $\theta_1 + \theta_2 = \text{const} \equiv C$, what is the relation between θ_1 and θ_2 , so that $\Delta\tau$ has its greatest value?

The straightforward solution is

$$\begin{aligned} \Delta\tau &= 2p [\sinh C - \sinh \theta_1 - \sinh(C - \theta_1)] \\ \frac{d(\Delta\tau)}{d\theta_1} &\equiv -\cosh \theta_1 + \cosh(C - \theta_1) = 0 \Rightarrow \theta_1 = C/2 \equiv \theta_2 \\ \left. \frac{d^2(\Delta\tau)}{d\theta_1^2} \right|_{\theta_1=C/2} &\equiv -\sinh(C/2) < 0 \end{aligned} \quad (14)$$

We have obtained the ‘‘intuitive Euclidean’’ solution that the greatest difference between the elapsed times, i.e., the shortest proper-time for the moving twin, is obtained for $\theta_1 = \theta_2$. For these value Eq. (11) corresponds to the well known solution [1]

$$\tau_{\overline{OR}} = \tau_{(\overline{OT}+\overline{TR})} \cosh \theta_1 \equiv \frac{\tau_{(\overline{OT}+\overline{TR})}}{\sqrt{1-v^2}}. \quad (15)$$

Now we give a geometrical interpretation of this problem. From Eq. (10) we know that if $\theta_1 + \theta_2 = C$, θ_3 is constant too, then the posed problem is equivalent to: what can be the position of the vertex T if the starting and final points and the angle θ_3 are given?

The problem is equivalent to have, in a triangle, a side and the opposite angle. In an equivalent problem in Euclidean geometry we know at once that the vertex T does move on a circle arc. Then, from the established correspondence of circles in Euclidean geometry to equilateral hyperbolas in pseudo-Euclidean geometry, we have that in the present space-time problem the vertex T will move on an arc of an equilateral hyperbola.

Now let us generalize the twin paradox to the case in which both twins change their state of motion: their motions start in O , both twins move on (different) straight-lines and cross again in R . The graphical representation is given by a quadrilateral figure and we call T and T' the other two vertices. Since a hyperbolic rotation of the triangle does not change the angles and the side lengths [5, 10], we can rotate the figure so that the vertex R lies on the t axis (see Fig. 1). The problem can be considered as a duplicate of the previous one in the sense that we can compare the proper times of both twins with the side \overline{OR} . If we indicate by $(')$ the quantities referred to the triangle under the t axis, we apply Eq. (13) twice and obtain $\Delta\tau - \Delta\tau'$ for every specific example.

In particular if we have $\theta_1 + \theta_2 = \theta'_1 + \theta'_2 = C$, from the result of Eq. (14) it follows that the youngest twin is the one for which θ_1 and θ_2 are closer to $C/2$.

3.2 Inertial and accelerated motions

Now we consider some “more realistic” examples in which uniformly accelerated motions are taken into account. The geometrical representation of a motion with constant acceleration is given by an arm of an equilateral hyperbola with the semi-diameter p linked to the acceleration a by the relation $p^{-1} = a$ ([15] p. 58, [16] p. 166, [24]).

Obviously, the geometrical representation of a motion with non-uniform acceleration is given by a curve which is the envelope of the equilateral hyperbolas corresponding to the instantaneous accelerations. Or, vice versa, we can construct in every point of a curve an “osculating hyperbola” which has the same properties of the osculating circle in Euclidean geometry. In fact the semi-diameter of these hyperbolas is linked to the second derivative with respect to the line element ([23] § 3.3) as the radius of osculating circles in Euclidean geometry.

We also indicate by $C \equiv (t_C, x_C)$ its center and with θ a parameter that, from a geometrical point of view, represents a hyperbolic angle measured with respect to an axis passing trough C and parallel to x axis [6, 10].

Then its equation, in parametric form, is

$$\mathcal{I} \equiv \begin{cases} t = t_C \pm p \sinh \theta \\ x = x_C \pm p \cosh \theta \end{cases} \quad \text{for } -\infty < \theta < \infty, \quad (16)$$

where the $+$ sign refers to the upper arm of the equilateral hyperbola and the $-$ sign to the lower one.

We also have

$$dx = \pm p \sinh \theta d\theta, \quad dt = \pm p \cosh \theta d\theta \quad (17)$$

and the proper time on the hyperbola

$$\tau_{\mathcal{I}} = \int_{\theta_1}^{\theta_2} \sqrt{dt^2 - dx^2} \equiv \int_{\theta_1}^{\theta_2} p d\theta \equiv p (\theta_2 - \theta_1). \quad (18)$$

This relation states the link between the proper time, the acceleration, and the hyperbolic angle and also shows that hyperbolic angles are given by the ratio between the “lengths” of the hyperbola arcs and the semidiameter as the circular angles in Euclidean trigonometry are given by the ratio between circle arcs and radius. Moreover, as in Euclidean geometry, the magnitude of hyperbolic angles is equal to twice the area of the hyperbolic sector [10] and, taking into account that the “area” is the same quantity in Euclidean and pseudo-Euclidean geometries, it can be calculated in a simple Euclidean way ([5] p. 183).

In point P , determined by $\theta = \theta_1$, the velocity is given by $v \equiv dx/dt = \tanh \theta_1$ and the straight-line tangent to the hyperbola for $\theta = \theta_1$ is given by [10]:

$$\begin{aligned} x - (x_C \pm p \cosh \theta_1) &= \tanh \theta_1 [t - (t_C \pm p \sinh \theta_1)] \Rightarrow \\ x \cosh \theta_1 - t \sinh \theta_1 &= x_C \cosh \theta_1 - t_C \sinh \theta_1 \mp p \end{aligned} \quad (19)$$

From this equation we see that θ_1 also represents the hyperbolic angle of the tangent to the hyperbola with the t axis. This last property means that semi-diameter \overline{CP} is pseudo-orthogonal to the tangent in P (see also Fig. 3)[25]. This property corresponds, in Euclidean counter-part, to the well known property of the circle where the radius is orthogonal to the tangent-line.

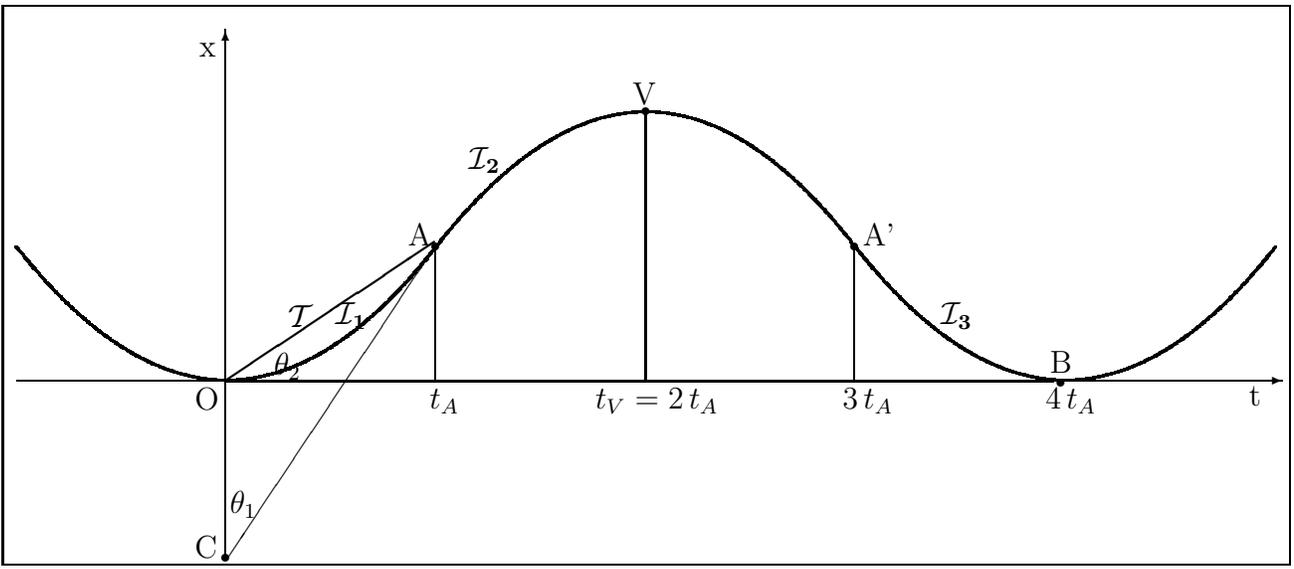


Figure 2: The motions of example 3.2.1

3.2.1 First example

We start with the following example in which the first twin after some accelerated motions returns to the starting point with vanishing velocity. The problem is represented in Fig. (2).

The first twin (I) starts with a constant accelerated motion with acceleration p^{-1} (indicated by \mathcal{I}_1) from O to A and then a constant decelerated (p^{-1}) motion up to V and then accelerated with reversed velocity up to A' (\mathcal{I}_2), then another decelerated motion (\mathcal{I}_3) as \mathcal{I}_1 up to $B \equiv (4t_A, 0)$;

the second twin (II) moves with a uniform motion (\mathcal{I}_1) that, without loss of generality, can be represented as stationary in the point $x = 0$.

Solution. The equilateral hyperbola \mathcal{I}_1 has its center in $C \equiv (0, -p)$. Then we have

$$\mathcal{I}_1 \equiv \begin{cases} t = p \sinh \theta \\ x = p (\cosh \theta - 1) \end{cases} \quad \text{for } 0 < \theta < \theta_1. \quad (20)$$

We also have $A \equiv (p \sinh \theta_1, p \cosh \theta_1 - p)$.

The simmetry of the problem indicates that for both twins the total elapsed times are four times the elapsed times of the first motion.

The proper time of twin I is obtained from Eq. (18)

$$\tau_I \equiv 4 \tau_{\mathcal{I}_1} = 4 \int_0^{\theta_1} \sqrt{dt^2 - dx^2} \equiv 4 \int_0^{\theta_1} p d\theta \equiv 4p\theta_1, \quad (21)$$

the proper time of twin II is

$$\tau_{II} \equiv 4t_A = 4p \sinh \theta_1. \quad (22)$$

The difference between the elapsed times is $\Delta\tau = 4p(\sinh \theta_1 - \theta_1)$, and their ratio is

$$\frac{\tau_I}{\tau_{II}} = \frac{\theta_1}{\sinh \theta_1}. \quad (23)$$

For $\theta_1 \equiv \tanh^{-1} v \ll 1$ [26] we have $\Delta\tau \simeq 0$, and for $\theta \gg 1 \Rightarrow \sinh \theta \propto \exp[\theta]$: *The proper time for the accelerated motions is linear in θ and the stationary (inertial) is exponential in θ .*

Now we show that the same relation between uniform and accelerated motion holds if we compare the motion on the side \overline{OA} with the motion on hyperbola \mathcal{I}_1 , and this allows us to give a simple ‘‘Euclidean’’ interpretation.

Let us call θ_2 the hyperbolic angle between straight-line OA and t axis; the equation of straight-line OA is

$$\mathcal{T} \equiv \{x = t \tanh \theta_2\} \quad (24)$$

and we calculate θ_2 imposing that this straight-line crosses the hyperbola of Eq. (20) for $\theta = \theta_1$. By substituting Eq. (24) in Eq. (20), we have

$$\begin{cases} t = p \sinh \theta_1 \\ t \tanh \theta_2 = p (\cosh \theta_1 - 1) \end{cases} \Rightarrow \frac{\sinh \theta_2}{\cosh \theta_2} = \frac{\cosh \theta_1 - 1}{\sinh \theta_1} \Rightarrow \theta_1 = 2 \theta_2, \quad (25)$$

i.e., the central angle is twice the hyperbola angle on the same chord [10]. Then we have

$$\overline{OA} = \frac{\overline{Ot_A}}{\cosh \theta_2} \equiv \frac{p \sinh \theta_1}{\cosh \theta_2} \equiv 2p \sinh \theta_2 \quad (26)$$

and taking into account the proper time on the hyperbola (Eq. 18), we obtain

$$\frac{\tau_{\mathcal{I}}}{\tau_{\mathcal{T}}} = \frac{\theta_2}{\sinh \theta_2}. \quad (27)$$

This relation is a general one and it is not surprising since it derives from the correspondence (see sec. 2.2) between Euclidean and pseudo-Euclidean geometries. In Euclidean geometry it represents the ratio between the length of a circle arc and its chord.

3.2.2 Second example

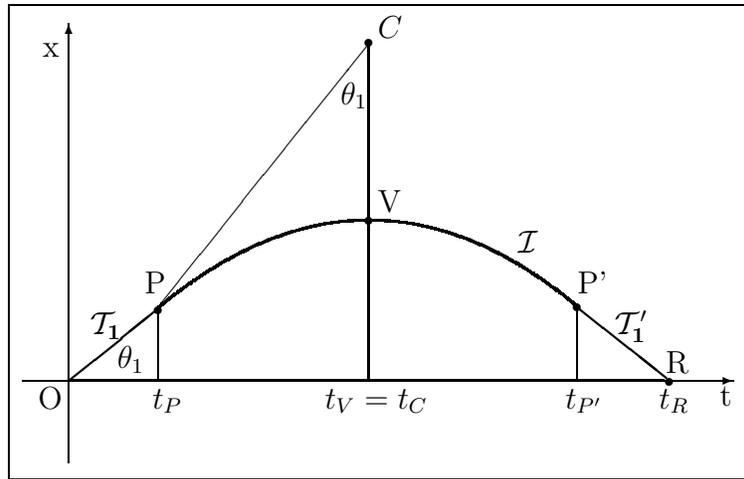


Figure 3: The motions of example 3.2.2

Now we consider a problem that allows us to connect the two sides of the triangle of Fig. (1) by means of an equilateral hyperbola, i.e., to consider the decelerated and accelerated motions too.

Twin I moves from $O \equiv (0, 0)$ to $P \equiv (p \sinh \theta_1, p \cosh \theta_1 - p)$ with a uniform motion, indicated as \mathcal{T}_1 , then goes on with a constant decelerated motion up to V and then accelerates with reversed velocity up to P' , where he has the same velocity as the initial one, and moves again with uniform velocity up to $R \equiv (t_R, 0)$ (\mathcal{T}'_1).

The second twin (II) moves with a uniform motion (\mathcal{T}_2) which, without loss of generality, can be represented as stationary in the point $x = 0$.

Solution. A mathematical formalization can be the following: let us consider the decelerated and accelerated motions that can be represented by the equilateral hyperbola of Eq. (16) for $-\theta_1 < \theta < \theta_1$ and the tangent to the hyperbola for $\theta = \theta_1$ as given by Eq. (19). This straight-line represents the motion \mathcal{T}_1 if it passes through O . This happens if center of the hyperbola $C \equiv (x_C, t_C)$ lies on the straight-line $x_C \cosh \theta_1 - t_C \sinh \theta_1 - p = 0$, where t_C is given by Eq. (16): $t_C = t_P + p \sinh \theta_1$. If we write down straight-line (19) in parametric form

$$\mathcal{T}_1 \equiv \begin{cases} t = \tau \cosh \theta_1 \\ x = \tau \sinh \theta_1 \end{cases}, \quad (28)$$

where τ is the proper time on the straight-line, we have at the end of the uniform motion $P \equiv (t_P, x_P)$, with $t_P = \tau \cosh \theta_1$.

Then the proper time for twin I is $\tau_{\overline{OP}} = \tau$, and from P to the vertex of the hyperbola $\tau_{\mathcal{I}} = p \theta_1$. The proper times of the other lines are a duplicate of these ones.

For twin II we have: $\tau_{II} \equiv 2 t_C = 2 (\tau \cosh \theta_1 + p \sinh \theta_1)$.

Then we have

$$\Delta \tau = 2 [\tau (\cosh \theta_1 - 1) + p (\sinh \theta_1 - \theta_1)]. \quad (29)$$

The proper time on this rounded off triangle is greater than the one on the triangle, as we shall better see in the next example.

The physical interpretation is that the velocity on the hyperbola arc is less than the one on straight-lines OT, TR of Fig. (1). From a geometrical point of view, it is a consequence of the reverse triangle inequality or, in a more general way, we can say that the geodesic lines between two given points (straight-lines) are the longest lines.

3.2.3 Third example

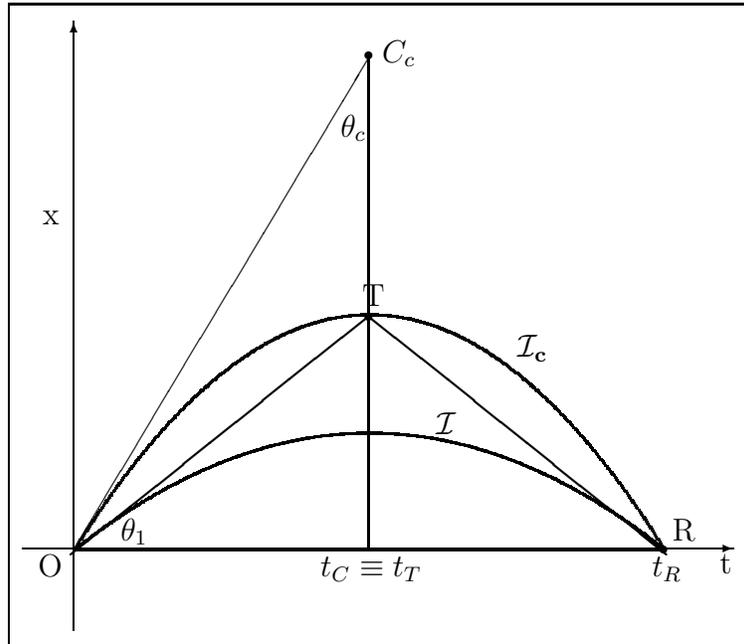


Figure 4: The motions of example 3.2.3

In the following example we consider the motion on the upper triangle of Fig. 1 with sides $\overline{OT} = \overline{TR}$ and on the following equilateral hyperbolas

1) \mathcal{I} tangent in O and in R to sides \overline{OT} and \overline{TR} , respectively

2) \mathcal{I}_c circumscribed to triangle $\overset{\Delta}{OTR}$.

In this example we can also note a formalization of the reverse triangle inequality ([23] p. 130). In fact, we shall see that as shorter the lines (trajectories) are in a Euclidean representation, so longer they are in the space-time geometry.

Solution. Side \overline{OT} lies on the straight-line represented by the equation

$$x \cosh \theta_1 - t \sinh \theta_1 = 0. \quad (30)$$

Hyperbola \mathcal{I} is obtained requiring that it is tangent to straight-line (30) in O . We obtain from Eqs. (16, 19) $t_C = p \sinh \theta_1$, $x_C = p \cosh \theta_1$ and, from the definitions of hyperbolic trigonometry, $\overline{OT} = t_C / \cosh \theta_1 \equiv p \tanh \theta_1$.

Let us consider hyperbola \mathcal{I}_c . Its vertex is T and its semi-diameter is given by Eq. (8): $p_c = \overline{OT} / (2 \sinh \theta_1) \equiv p / (2 \cosh \theta_1)$. If we call C_c its center and $2\theta_c$ angle $\overline{OC_cR}$, we note that θ_c is a central angle of chord \overline{OT} while θ_1 is an hyperbola angle on equal chord \overline{TR} . Then, as it has been shown in example 3.2.1, we have $\theta_c = 2\theta_1$.

Now let us calculate the lengths (proper times) for the motions.

As to the hyperbolas, from Eq. (18), we have:

the length of arc of hyperbola \mathcal{I} between O and R is given by $\tau_{\mathcal{I}} = 2p\theta_1$,

the length of arc of \mathcal{I}_c from O and R is given by $\tau_{\mathcal{I}_c} = 2p_c\theta_c \equiv 2p\theta_1 / \cosh \theta_1$.

For the lengths of the segments we have:

$\overline{OR} \equiv 2t_T = 2p \sinh \theta_1$, and from Eq. (30) it follows $T \equiv (p \sinh \theta_1, p \sinh \theta_1 \tanh \theta_1)$, so $\overline{OT} = \overline{TR} = p \tanh \theta_1$.

Then we have the following relations:

$$\overline{OR} \equiv 2p \sinh \theta_1 > \text{arc}(\mathcal{I}) \equiv 2p\theta_1 > \overline{OT} + \overline{TR} \equiv 2p \tanh \theta_1 > \text{arc}(\mathcal{I}_c) \equiv 2p\theta_1 / \cosh \theta_1. \quad (31)$$

We also observe that \overline{OR} is a chord of \mathcal{I} , \overline{OT} is a chord of \mathcal{I}_c and their ratios are the one given by Eq. (27):

$$\frac{\tau_{\overline{OR}}}{\tau_{\mathcal{I}}} = \frac{\tau_{\overline{OT}}}{\tau_{\mathcal{I}_c}} \equiv \frac{\sinh \theta_1}{\theta_1}. \quad (32)$$

As a corollary of this example we consider the following one: given side $\overline{OR} = \tau$ (proper time of the stationary twin) what is the proper time of twin I moving on an equilateral hyperbola, as a function of rocket acceleration p^{-1} ?

Solution. From hyperbolic motion of Eq. (16) we have $t = \tau/2 - p \sinh \theta$ and for $t = 0$ we obtain $\theta_1 \Rightarrow 2p \sinh \theta_1 = \tau$, and for relativistic motions ($\theta_1 \gg 1$) we obtain

$$\tau \simeq p \exp[\theta_1] \Rightarrow \theta_1 \simeq \ln \frac{\tau}{p}. \quad (33)$$

Then from relation (18) $\tau_{\mathcal{I}} \equiv 2p\theta_1 = 2p \ln[\tau/p] \xrightarrow{p \rightarrow 0} 0$.

As acceleration p^{-1} does increase, proper time $\tau_{\mathcal{I}}$ can be as less as we want ([16] p. 167).

3.2.4 Fourth example

We conclude with a more general example in which both twins have a uniform and accelerated motion.

First twin (I) starts with a constant accelerated motion and then goes on with a uniform motion, second twin (II) starts with a uniform motion and then goes on with a constant accelerated motion.

Solution. We can represent this problem in the t, x plane in the following way:

I starts from point $O \equiv (0, 0)$ with an acceleration given by p^{-1} (\mathcal{I}_1) up to point $A \equiv (p \sinh \theta_1; p \cosh \theta_1 - p)$, then goes on with a uniform motion (\mathcal{T}_1) up to time t_3 (point B).

II starts from point $O \equiv (0, 0)$ with a uniform motion (\mathcal{I}_2), (stationary in the point $x = 0$) up to point C , in a time $t_2 = \alpha p \sinh \theta_1$ that we have written proportional to t_A . Then goes on with an accelerated motion (\mathcal{I}_2), with the same acceleration p^{-1} up to crossing the trajectory of I at time t_3 . The analytical representation of \mathcal{I}_1 is given by Eq. (20). \mathcal{I}_2 is represented by

$$\mathcal{I}_2 \equiv \begin{cases} t = p(\alpha \sinh \theta_1 + \sinh \theta) \\ x = p(\cosh \theta - 1) \end{cases} \text{ for } 0 < \theta < \theta_2, \quad (34)$$

where θ_2 represents the value of the hyperbolic angle in crossing point B between \mathcal{I}_2 and \mathcal{T}_1 . \mathcal{T}_1 is given by the straight-line tangent to \mathcal{I}_1 in θ_1 :

$$\mathcal{T}_1 \equiv \{x \cosh \theta_1 - t \sinh \theta_1 = p(1 - \cosh \theta_1)\}. \quad (35)$$

From Eqs. (34) and (35) we calculate the crossing point between \mathcal{I}_2 and \mathcal{T}_1 . We have[27]

$$\cosh(\theta_2 - \theta_1) = \alpha \sinh^2 \theta_1 + 1. \quad (36)$$

Let us calculate the proper times.

The proper times relative to the accelerated motions are obtained from Eq. (18): $\tau_{\mathcal{I}_1} = p \theta_1$, $\tau_{\mathcal{I}_2} = p \theta_2$. The proper time relative to \mathcal{T}_2 is given by $t_2 = \alpha p \sinh \theta_1$. On straight-line \mathcal{T}_1 , between points A and $B \equiv (p \alpha \sinh \theta_1 + p \sinh \theta_2, p \cosh \theta_2 - p)$, the proper time is obtained by means of hyperbolic trigonometry [10]

$$\tau_{\mathcal{T}_1} \equiv \overline{AB} = (x_B - x_A) / \sinh \theta_1 \equiv p(\cosh \theta_2 - \cosh \theta_1) / \sinh \theta_1. \quad (37)$$

Then the complete proper-times of the twins are

$$\tau_I = p[\theta_1 + (\cosh \theta_2 - \cosh \theta_1) / \sinh \theta_1], \quad \tau_{II} = p(\alpha \sinh \theta_1 + \theta_2). \quad (38)$$

Let us consider relativistic velocities ($v = \tanh \theta_i \simeq 1 \Rightarrow \theta_1, \theta_2 \gg 1$); in this case we can approximate the hyperbolic functions in Eqs. (36, 38) with the positive exponential term and, for $\alpha \neq 0$, we obtain from Eq. (36): $\exp[\theta_2 - \theta_1] \simeq \alpha \exp[2\theta_1]/2$, and from Eqs. (38)

$$\tau_I \simeq p(\theta_1 + \exp[\theta_2 - \theta_1]) \simeq p(\theta_1 + \alpha \exp[2\theta_1]/2), \quad \tau_{II} \simeq p(\alpha \exp[\theta_1]/2 + \theta_2). \quad (39)$$

The greatest contributions to the proper times are given by the exponential terms that derive from the uniform motions. If we neglect the linear terms with respect to the exponential ones, we obtain a ratio of the proper times *independent of the $\alpha \neq 0$ value*

$$\tau_I \simeq \tau_{II} \exp[\theta_1] \quad (40)$$

The twin that moves for a shorter time with uniform motion has the shortest proper time[28].

A simplified version (an inertial and an accelerated motion between two points) is given in [16] (exercise 6.3 p.167) and it is considered just as a consequence of the reverse triangle inequality.

With regard to the result of this example we could ask: how is it possible that a uniform motion close to a light-line is the longer one? We can answer this question by a glance at Eq. (37). In fact in this equation the denominator $\sinh \theta_1 \gg 1$ takes into account that the motion is close to a light-line, but in the numerator $\cosh \theta_2 \gg \cosh \theta_1$ indicates that crossing point B is so far that its contribution is the determining term of the result we have obtained.

4 Conclusions

As we know, the twin paradox spread far and wide having been considered the most striking exemplification of the space-time “strangeness” of Einstein’s theory of special relativity. What we have striven to show is that hyperbolic trigonometry supplies us with an easy tool by which one can deal with any kinematic problem in the context of special relativity.

In fact, if we consider Einstein’s theory of special relativity as a logical-deductive construction based on the two postulates of the constancy of light’s velocity and the equivalence of all inertial reference frames for establishing physical laws, the hyperbolic space-time is the right mathematical structure inside which any problem must be dealt with.

As we have seen, the use of hyperbolic trigonometry allows us to obtain the *quantitative solution* of any problem and dispels all doubts regarding the role of acceleration in the flow of time.

Finally, we remark that the application of hyperbolic trigonometry to relativistic space-time results to be a “Euclidean way” of dealing with pseudo-Euclidean spaces.

A The formalization of Euclidean and pseudo-Euclidean trigonometries by means of complex or hyperbolic numbers

For greater convenience of the reader we report a short exposition of paper [10]

A.1 Rotation invariants in Euclidean plane

Euclid’s geometry studies the figure properties that do not depend on their position in a plane. If these figures are represented in a Cartesian plane we can say, in group language, that Euclid’s geometry studies the invariant properties by coordinate axes roto-translations. It is well known that these properties can be expressed by complex numbers. Let us consider Gauss-Argand’s complex plane where a vector is represented by $v = x + iy$. The axes rotation of an angle α transforms this vector in the new vector $v' \equiv v \exp[i\alpha]$. Therefore we can promptly verify that the quantity (as it is usually done we call $\bar{v} = x - iy$) $|v'|^2 \equiv v' \bar{v}' = v \exp[i\alpha] \bar{v} \exp[-i\alpha] \equiv |v|^2$ is invariant by axes rotation. In a similar way we find two invariant quantities related to any couple of vectors.[29]

If we consider two vectors: $v_1 = x_1 + iy_1$, $v_2 = x_2 + iy_2$; we have that *the real and the imaginary part of the product $v_2 \bar{v}_1$ are invariant by axes rotation*. In fact $v_2 \bar{v}'_1 = v_2 \exp[i\alpha] \bar{v}_1 \exp[-i\alpha] \equiv v_2 \bar{v}_1$. Now we will see that these two invariant quantities allow an operative definition of trigonometric functions. Let us represent the two vectors in polar coordinates: $v_1 \equiv \rho_1 \exp[i\phi_1]$, $v_2 \equiv \rho_2 \exp[i\phi_2]$. Consequently we have:

$$v_2 \bar{v}_1 = \rho_1 \rho_2 \exp[i(\phi_2 - \phi_1)] \equiv \rho_1 \rho_2 [\cos(\phi_2 - \phi_1) + i \sin(\phi_2 - \phi_1)]. \quad (41)$$

As it is well known the resulting real part of this product represents the scalar product, while the “imaginary” part represents the modulus of the vector product, i.e., the area of the parallelogram defined by the two vectors.

The two invariant quantities of Eq. (41) allow an operative definition of trigonometric functions. In fact in Cartesian coordinates we have:

$$v_2 \bar{v}_1 = (x_2 + iy_2)(x_1 - iy_1) \equiv x_1 x_2 + y_1 y_2 + i(x_1 y_2 - x_2 y_1), \quad (42)$$

and by using Eqs. (41) and (42) we obtain:

$$\cos(\phi_2 - \phi_1) = \frac{x_1 x_2 + y_1 y_2}{\rho_1 \rho_2}; \quad \sin(\phi_2 - \phi_1) = \frac{x_1 y_2 - x_2 y_1}{\rho_1 \rho_2} \quad (43)$$

We know that the theorems of Euclid's trigonometry are usually obtained by following a geometric approach. Now by using the Cartesian representation of trigonometric functions given by Eqs. (43), it is straightforward to control that the trigonometry theorems are simple identities. In fact let us define a triangle in a Cartesian plane by its vertices $P_n \equiv (x_n, y_n)$: from the coordinates of these point we obtain the side lengths and, from Eqs. (43) the trigonometric functions. By these definitions it is easy to control the identities defined by the trigonometry theorems.

A.2 Hyperbolic rotation invariants in pseudo-Euclidean plane

By analogy with Euclid's trigonometry approach summarised in appendix (A.1), we can say that *pseudo-Euclidean plane geometry* studies the properties that are invariant by Lorentz transformations (Lorentz-Poincarè group of special relativity) corresponding to hyperbolic rotation as exposed in [8]. We show afterwards, how these properties can be represented by hyperbolic numbers.

Let us define in the hyperbolic plane a hyperbolic vector from the origin to the point $P \equiv (t, x)$, as $v = t + h x$ and consider a hyperbolic rotation of an angle θ that, from a physical point of view, means a Lorentz transformation with a velocity given by $V = \tanh^{-1} \theta$ [6, 15]. From this transformation the vector v become $v' \equiv v \exp[h \theta]$. Therefore we can readily verify that the quantity:

$$|v'|^2 \equiv v' \tilde{v}' = v \exp[h \theta] \tilde{v} \exp[-h \theta] \equiv |v|^2 \quad (44)$$

is invariant for hyperbolic rotation. In a similar way we can find two invariants related to any couple of vectors. Let us consider two vectors $v_1 = t_1 + h x_1$ and $v_2 = t_2 + h x_2$: we have that *the real and the "hyperbolic" parts of the product $v_2 \tilde{v}_1$ are invariant by hyperbolic rotation*.

In fact $v_2' \tilde{v}_1' = v_2 \exp[h \alpha] \tilde{v}_1 \exp[-h \alpha] \equiv v_2 \tilde{v}_1$. These two invariants allow an operative definition of the hyperbolic trigonometric functions. To show this let us suppose that $|t_1| > |x_1|$, $|t_2| > |x_2|$ and $t_1, t_2 > 0$, and let us represent the two vectors in hyperbolic polar form (4): $v_1 = \rho_1 \exp[h \theta_1]$, $v_2 = \rho_2 \exp[h \theta_2]$. Consequently we have

$$v_2 \tilde{v}_1 \equiv \rho_1 \rho_2 \exp[h(\theta_2 - \theta_1)] \equiv \rho_1 \rho_2 [\cosh(\theta_2 - \theta_1) + h \sinh(\theta_2 - \theta_1)]. \quad (45)$$

As shown in appendix (A.1), for Euclidean plane the real part of the vector product represents the scalar product, while the imaginary part represents the area of the parallelogram defined by the two vectors. In pseudo-Euclidean plane, as we know from differential geometry [30], the real part is still the scalar product; as far as the hyperbolic part is concerned, we see in subsection (A.3) that it can be considered as a *pseudo-Euclidean area* [5].

In Cartesian coordinates we have:

$$v_2 \tilde{v}_1 = (t_2 + h x_2)(t_1 - h x_1) \equiv t_1 t_2 - x_1 x_2 + h(t_1 x_2 - t_2 x_1). \quad (46)$$

By using Eqs. (45) and (46) we obtain:

$$\cosh(\theta_2 - \theta_1) = \frac{t_1 t_2 - x_1 x_2}{\rho_1 \rho_2} \equiv \frac{t_1 t_2 - x_1 x_2}{\sqrt{|(t_2^2 - x_2^2)|} \cdot \sqrt{|(t_1^2 - x_1^2)|}} \quad (47)$$

$$\sinh(\theta_2 - \theta_1) = \frac{t_1 x_2 - t_2 x_1}{\rho_1 \rho_2} \equiv \frac{t_1 x_2 - t_2 x_1}{\sqrt{|(t_2^2 - x_2^2)|} \cdot \sqrt{|(t_1^2 - x_1^2)|}}. \quad (48)$$

If we put $v_1 \equiv (1; 0)$ and $\theta_2, t_2, x_2 \rightarrow \theta, t, x$ then Eqs. (47), (48) can be rewritten in the form:

$$\cosh \theta = \frac{t}{\sqrt{|t^2 - x^2|}}; \quad \sinh \theta = \frac{x}{\sqrt{|t^2 - x^2|}}. \quad (49)$$

The classic hyperbolic functions are defined for $t, x \in \mathbf{R}$. Now we can observe that expressions in Eq. (49) are valid for $\{t, x \in \mathbf{R} | t \neq \pm x\}$ so they allow to extend the hyperbolic functions in the complete t, x plane. This extension is the same as that already proposed in [6]. These **extended hyperbolic functions** have been denoted with \cosh_e, \sinh_e in the paper [10]. In tab. (1) the relations between \cosh_e, \sinh_e and traditional hyperbolic functions are reported.

Table 1: Relations between functions \cosh_e, \sinh_e obtained from Eq. (49) and classic hyperbolic functions. The hyperbolic angle θ in the last four columns is calculated referring to semi- axes $t, -t, x, -x$, respectively.

	$ t > x $		$ t < x $	
	$(Rs), t > 0$	$(Ls), t < 0$	$(Us), x > 0$	$(Ds), x < 0$
$\cosh_e \theta =$	$\cosh \theta$	$-\cosh \theta$	$\sinh \theta$	$-\sinh \theta$
$\sinh_e \theta =$	$\sinh \theta$	$-\sinh \theta$	$\cosh \theta$	$-\cosh \theta$

The complete representation of the extended hyperbolic functions can be obtained by giving to t, x all the values on the circle $t = \cos \phi, x = \sin \phi$ for $0 \leq \phi < 2\pi$: in this way Eq. (49) become:

$$\cosh_e \theta = \frac{\cos \phi}{\sqrt{|\cos 2\phi|}} \equiv \frac{1}{\sqrt{|1 - \tan^2 \phi|}}; \quad \sinh_e \theta = \frac{\sin \phi}{\sqrt{|\cos 2\phi|}} \equiv \frac{\tan \phi}{\sqrt{|1 - \tan^2 \phi|}}. \quad (50)$$

These equations represent a bijective mapping between the points on unit circle (specified by ϕ) and the points on unit hyperbolas (specified by θ). From a geometrical point of view Eq. (50) represent the projection, from the coordinate axes origin, of the unit circle on the unit hyperbolas. From the definitions of extended hyperbolic trigonometric functions of Eqs. (47, 48), we can state for triangle in pseudo-Euclidean plane, exactly the same relations between sides and angles as the ones that hold for Euclidean triangles [10].

A.3 Trigonometry

In fact let us consider a triangle in pseudo-Euclidean plane with no sides parallel to axes bisectors: let us call $P_n \equiv (x_n, y_n) \ n = i, j, k \ | \ i \neq j \neq k$ the vertices, θ_n the hyperbolic angles. The square hyperbolic length of the side opposite to vertex P_i is defined by Eq. (5):

$$D_i \equiv D_{j,k} = (z_j - z_k)(\tilde{z}_j - \tilde{z}_k) \text{ and } d_i = \sqrt{|D_i|}. \quad (51)$$

as pointed out before D_i must be taken with its sign.

Following the conventions of Euclidean trigonometry we associate to the sides three vectors oriented from $P_1 \rightarrow P_2; P_1 \rightarrow P_3; P_2 \rightarrow P_3$.

From (47), (48) and taking into account the sides orientation as done in Euclidean trigonometry, we obtain:

$$\begin{aligned} \cosh_e \theta_1 &= \frac{(x_2 - x_1)(x_3 - x_1) - (y_2 - y_1)(y_3 - y_1)}{d_2 d_3}; \quad \sinh_e \theta_1 = \frac{(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)}{d_2 d_3} \\ \cosh_e \theta_2 &= \frac{(y_3 - y_2)(y_2 - y_1) - (x_3 - x_2)(x_2 - x_1)}{d_1 d_2}; \quad \sinh_e \theta_2 = \frac{(x_2 - x_1)(y_3 - y_2) - (y_2 - y_1)(x_3 - x_2)}{d_1 d_2} \\ \cosh_e \theta_3 &= \frac{(x_3 - x_2)(x_3 - x_1) - (y_3 - y_2)(y_3 - y_1)}{d_1 d_3}; \quad \sinh_e \theta_3 = \frac{(x_3 - x_1)(y_3 - y_2) - (y_3 - y_1)(x_3 - x_2)}{d_1 d_3} \end{aligned}$$

It is straightforward to verify that all the functions $\sinh_e \theta_n$ have the same numerator. If we call this numerator:

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 2S \quad (52)$$

we can write:

$$2S = d_2 d_3 \sinh_e \theta_1 = d_1 d_3 \sinh_e \theta_2 = d_1 d_2 \sinh_e \theta_3. \quad (53)$$

In Euclidean geometry a quantity equivalent to S represents the area of the triangle. In pseudo-Euclidean geometry S is still *an invariant quantity linked to the triangle*. For this reason it is appropriate to call S *pseudo-Euclidean area* [5].

We note that the expression of area (Eq. 52), in terms of vertices coordinates, is exactly the same as in Euclidean geometry (Gauss formula for a polygon area applied to a triangle).

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- [3] Langevin's address to the Congress of Bologna was published on *Scientia* **10**, 31-34 (1911). As reported by Miller [1], the popularisation of relativity theory for philosophers had an immediate impact which we can gauge from the comment of one of the philosophers present. Henry Bergson (1922) wrote:
"..it was Langevin's address to the Congress of Bologna on 10 April 1911 that first drew our attention to Einstein's ideas. We are aware of what all those interested in the theory of relativity owe to the works and teachings of Langevin.
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- [24] In [8] the hyperbolic motion is obtained as a straightforward consequence of the invariance of the wave equation with respect to Lorentz transformations.
- [25] We recall that two straight-line are pseudo-orthogonal if they are symmetric with respect to a couple of lines parallel to the axes bisectors [10, 15].
- [26] In the standard system of units this condition means $v \ll c$.
- [27] The following equation has an explicit solution for $\alpha = 2$. In fact we have
$$\cosh(\theta_2 - \theta_1) = 2 \sinh^2 \theta_1 + 1 \equiv \cosh 2 \theta_1 \Rightarrow \theta_2 = 3 \theta_1.$$
- [28] Since the total time is the same, shorter time with uniform motion means longer time with accelerated motion, so this result is that of Eq. (27)
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